A <u>monoid</u> is set w/ comm. binary operation, and identity element. Usually denoted $(+, 0)$ P

Given a field k have monoid algebra

$$k[P] = \bigoplus_{p \in P} k \chi^{p}$$

$$\chi^{p} \cdot \chi^{p'} = \chi^{p+p'}$$

<u>Ex</u>

$k[\mathbb{N}] \cong k[x]$

$k[\mathbb{N}^r] = k[x_1, \dots, x_r]$

Throughout we fix the notation:

$$N = \mathbb{Z}^n \qquad M = N^\vee = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$$

$$N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}^n \qquad M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}^n$$

Def

$\sigma \leq N_{\mathbb{R}}$ is a convex rational polyhedral cone

if it's of the form:

$$\sigma = \mathrm{Cone}(S) = \left\{ \sum_u \lambda_u u \;\middle|\; \lambda_u \geq 0 \right\}$$

$S \subseteq M$ finite.

- A face of a cone σ is $\tau := H_m \cap \sigma$ for some hyperplane $H_m = \{u \in N_{\mathbb{R}} \mid \langle u, m \rangle = 0\}$ s.t. $\sigma \subseteq H_m^+ = \{u \in N_{\mathbb{R}} \mid \langle u, m \rangle \geq 0\}$
  $m \in M_{\mathbb{R}}$

  vertex = dim 0   edge = dim 1
  facet = dim $\mathbb{R}^n - 1$

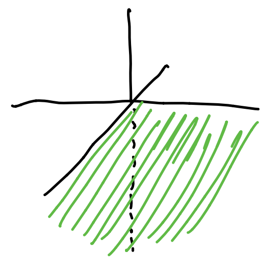
- A cone σ is strictly CRPC if it has $\{0\}$ as a face.

- The dual cone of σ is

$$\sigma^\vee = \{ m \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq 0 \quad \forall u \in \sigma \}$$

Ex

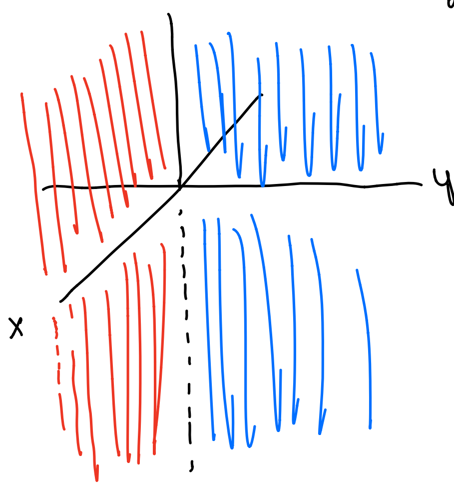
$\sigma = \mathrm{Cone}(e_1, e_2) \subseteq \mathbb{R}^2$



σ

Note σ is strictly____ but dual isn't

$\sigma^\vee$



$\sigma^\vee = \mathrm{Cone}(f_1, f_2, f_3, -f_3)$

this is not strictly convex.

Indeed it contains the z-axis

Hence $\{0\}$ can't be a face

(b/c of the supporting closed half space!)

$(y=0) \cap (x=0)$ positive part.

---

strongly convex rational poly. cones have unique generators in the following sense:

- let $\rho \leq \sigma$ be an edge. $\Rightarrow$ it's a ray departing from the origin. (faces of S.C.RPC are also SCRPC)

Since ρ rational too $\rho \cap N$ generated by a unique element.

called u_ρ ray generators

Prop

- A st. CRPC is generated by its ray generators.

More facts

- Gordon's theorem $\Longrightarrow$ $\sigma^\vee \cap M \subseteq M$ is finitely generated monoid (commutative addition, identity element) i.e. $\mathbb{N}m_1 \oplus \dots \oplus \mathbb{N}m_s \subseteq M$

- Finally $X_\sigma = \operatorname{Spec} k[\sigma^\vee \cap M]$ is the affine toric variety defined by the cone σ.

(why is it so? $k[x_1, \dots, x_s] \overset{\text{closed}}{\twoheadrightarrow} k[\sigma^\vee \cap M] \xrightarrow{\text{dominant.}} k[M]$
$x_i \longmapsto \chi^{m_i}$)

Ex

$\sigma = \operatorname{Cone}(e_1, e_2) \subseteq \mathbb{R}^2$

$\sigma^\vee = \operatorname{Cone}(e_1^\vee, e_2^\vee) \subseteq \mathbb{R}^2$

$$k[\sigma^\vee \cap \mathbb{Z}^2] = k[x, y]$$

Def Fans $\rightarrow$ Toric

A fan $\Sigma \subseteq N_{\mathbb{R}}$ is a finite collection of cones

$\sigma \subseteq N_{\mathbb{R}}$ s.t.

1. every $\sigma \in \Sigma$ is strongly convex rational polyhedral cone

2. $\forall \sigma \in \Sigma$ each face of σ is in Σ

3. $\sigma_1, \sigma_2 \in \Sigma \implies \sigma_1 \cap \sigma_2$ is a face of each. (and hence in Σ)

For each cone $\sigma \in \Sigma$ we get $\underset{X_\sigma}{\overset{\,}{\mathrm{Spec}\,k[\sigma^\vee \cap M]}}$ affine toric

If $\tau \leq \sigma$ ($\Leftrightarrow \exists m \in \sigma^\vee \cap M$ s.t. $\tau = H_m \cap \sigma$)

then $X_\tau = (X_\sigma)_{\chi^m}$ = the locus in X_σ where χ^m is non vanishing

Above we require $\sigma_1 \cap \sigma_2$ to be a face of each σ_1, σ_2.

One can find $m \in \mathrm{Relint}(\sigma_1^\vee \cap (-\sigma_2)^\vee)$ s.t.

$$(X_{\sigma_1})_{\chi^m} = X_\tau = (X_{\sigma_2})_{\chi^{-m}}$$

After all these gluings we get a separated toric variety.

Ex

$N = \mathbb{Z}$ $\quad M = \mathbb{Z}$

$N_{\mathbb{R}} = \mathbb{R}$

only cones are $\sigma_+ = [0, \infty)$

$\sigma_- = (-\infty, 0]$

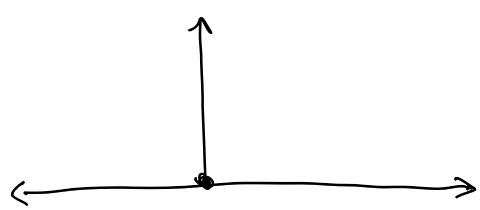
$\tau = \{0\} = \mathrm{Cone}(\emptyset)$

$\{\tau\} \longrightarrow \mathbb{C}^*$

$\{\sigma_-, \tau\}$, $\{\sigma_+, \tau\}$ both give $\mathbb{C}$

$\{\sigma_+, \sigma_-, \tau\}$ gives $\mathbb{P}^1$.

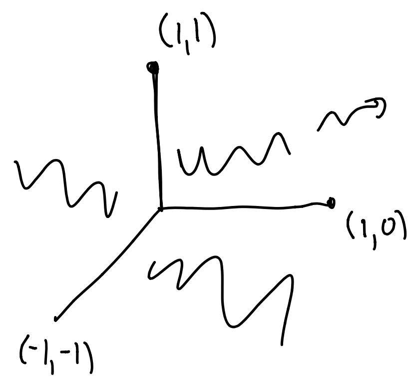
Ex



gives $\mathbb{P}^1 \times \mathbb{C}$

(product of fans above.)

Ex



$\rightsquigarrow \mathbb{P}^2$

Functoriality

$N_1 \xrightarrow{\phi} N_2$ lattices w/ group homo.

$\Sigma_i \subseteq (N_i)_{\mathbb{R}}$ fans

ϕ is a map of fans if $\forall \sigma_1 \in \Sigma_1 \ \exists \sigma_2 \in \Sigma_2$ s.t.

$\phi(\sigma_1) \subseteq \sigma_2$

$\Rightarrow \quad \phi^\vee(\sigma_2^\vee) \subseteq \sigma_1^\vee$

$\Rightarrow \quad k[\sigma_2^\vee \cap M_2] \longrightarrow k[\sigma_1^\vee \cap M_1]$

these maps are compatible for various σ_1 & so glue to

$$X_{\Sigma_1} \longrightarrow X_{\Sigma_2}$$

Divisors

Orbit-Cone says that

$$\{\text{rays } \rho \text{ of } \Sigma\} \longleftrightarrow \{O(\rho) \text{ codim } 1 \text{ orbits}\}$$

the closure $\overline{O(\rho)}$ is T_N-invariant prime divisor.

$= D_\rho$. Hence have a valuation $v_\rho : \mathbb{C}(X_\Sigma)^* \to \mathbb{Z}$

Recall a ray $\rho \in \Sigma$ has minimal generator $u_\rho \in \rho \cap N$.

Prop

w/ the above setup

$$v_\rho(\chi^m) = \langle m, u_\rho \rangle$$

$$\operatorname{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho$$

Def "Dual"

A lattice polytope is $P = \mathrm{Conv}(S)$ for $S \subseteq M$. finite.

Have faces, facets, vertices just as cones except this time defined by supporting affine hyperplanes.

$$F = H_{u,b} \cap P \qquad P \subseteq H^+_{u,b}$$

$$H_{u,b} = \{ m \in M_{\mathbb{R}} \mid \langle m, u \rangle = b \} \quad u \in N_{\mathbb{R}}$$

$$H^+_{u,b} = \{ m \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq b \}$$

P is full dimensional it nice presentation:

$$P = \{ m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq -a_F \quad \forall \text{ facets} \}$$

$u_F, a_F \in M$ unique.

$v \in P$ a vertex we have a cone $C_v = \mathrm{Cone}(P \cap M - v) \subseteq M_{\mathbb{R}}$

Have correspondence

$$\left\{ \begin{array}{l} Q \subseteq P \\ \text{faces} \\ \text{contain } v \end{array} \right\} \qquad \left\{ \begin{array}{l} \text{faces } Q_v \subseteq C_v \\ Q_v = \mathrm{Cone}(Q \cap M - v) \end{array} \right\}$$

$$Q \longmapsto Q_v$$

$(Q_v + v) \cap P \hookleftarrow Q_v$

bijections preserves dim, inclusion, intersections

We can construct a fan from this by writing:

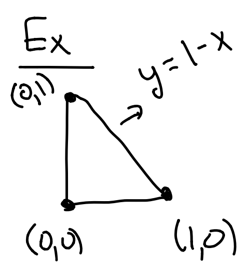
$$\sigma_Q = \text{Cone}(u_F \mid F \text{ facets containing } Q)$$
$$= Q_v^\vee$$

<u>Thm</u>

P full dim lattice polytope

$\Sigma_P = \{\sigma_Q \mid Q \leq P\}$ is a fan called normal fan

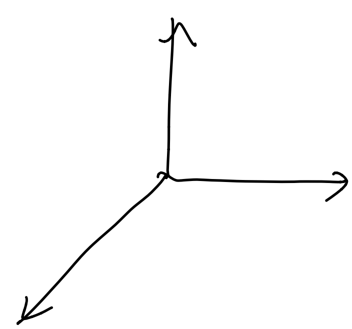
<u>Ex</u>



$\sigma_{(0,0)} = \text{Cone}(e_1, e_2)$

$\sigma_{(0,1)} = \text{Cone}(e_1, (-1,-1))$

$\sigma_{(1,0)} = \text{Cone}(e_2, (-1,-1))$



$\Downarrow$

$\mathbb{P}^2$

A **polyhedron** $P \subseteq M_{\mathbb{R}}$ is the intersection of finitely many closed half spaces

$$P = \{ m \in M_{\mathbb{R}} \mid \langle m_i, u_i \rangle \geq -a_i \quad i = 1, \ldots, s \}$$

Basic structure theorem says

$$P = Q + C$$

Q polytope

C polyhedral cone $\quad P$ as above $\quad C = \{ m \in M_{\mathbb{R}} \mid \langle m, u_i \rangle \geq 0 \ \ i = 1, \ldots s \}$

<u>lattice</u> polyhedron (WRT to M) if

(a) $\mathrm{rec}(P) := C$ is strongly convex rational polyhedral cone

$\Updownarrow$ to P having finitely many vertices

(b) vertices of P lie in M.

Similar to polytopes, we have supporting hyperplanes, vertices, faces, facets, Full dim lattice polytope has unique facet presentation:

stick with

$$P = \{m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq -a_F \quad \forall F \text{ facet}\}$$

$u_F \in N$ inward pointing normal.

Cone of P

$$\mathcal{C}(P) := \{(m, \lambda) \in M_{\mathbb{R}} \times \mathbb{R} \mid \langle m, u_F \rangle \geq -\lambda a_F \quad \forall F, \ \forall \lambda \geq 0\}$$

Say $(m, \lambda) \in \mathcal{C}(P)$ has height λ

Note when $\lambda > 0$ the slice of $\mathcal{C}(P)$ at height λ is λP

Lemma

P full dim lattice polyhedron in $M_{\mathbb{R}}$

$\operatorname{rec} P = C$ then $\mathcal{C}(P)$ is a strongly convex cone in $M_{\mathbb{R}} \times \mathbb{R}$

al

$$\mathcal{C}(P) \cap (M_{\mathbb{R}} \times \{0\}) = C = \operatorname{res}(P)$$

Polyhedron → Toric Variety

★ We build the normal fan of a polyhedron in exactly the same way as a polytope. Denote it by Σ_P

Prop

P lattice polyhedron with recession cone C

$$|\Sigma_P| = \bigcup_{\sigma \in \Sigma_P} \sigma = C^\vee$$

$\Longrightarrow X_{\Sigma_P}$ isnt complete (complete means $|\Sigma| = N_{\mathbb{R}}$)

Set $W = |\Sigma_P| \cap -|\Sigma_P| \subseteq |\Sigma_P|$

$|\Sigma_P|$ gives:

- $W \cap N \subseteq N$ sublattice + $N_P := N/N \cap W$ quotient lattice
- strongly convex rational cone $\sigma_P = |\Sigma_P|/W \subseteq N_{\mathbb{R}}/W = (N_P)_{\mathbb{R}}$
- affine toric variety U_{σ_P}

The projection

$$N \xrightarrow{\bar{\phi}} N_P$$

is compatible with the fans Σ_p, σ_p

And so we get toric morphism (since $\Phi(|\Sigma_p|) = \sigma_p$)

$$X_{\Sigma_p} \longrightarrow U_p$$

this is actually projective.

Note each affine piece of normal fan maps to U_p.

Def

A (lattice) polyhedral decomposition of a lattice polyhedron $\Delta \subseteq M_{\mathbb{R}}$ is a set $\mathcal{P}$ of (lattice) polyhedra in $M_{\mathbb{R}}$, called cells, s.t.

1. $\Delta = \bigcup_{\sigma \in \mathcal{P}} \sigma$
2. If $\sigma \in \mathcal{P}$ and $\tau \leq \sigma$ a face, then $\tau \in \mathcal{P}$
3. If $\sigma_1, \sigma_2 \in \mathcal{P} \Rightarrow \sigma_1 \cap \sigma_2$ is a face of both.

A affine PAL on Δ WRT $\mathcal{P}$ if its linear when restricted to each $\sigma \in \mathcal{P}$, and it's strictly convex WRT $\mathcal{P}$ if

1. Δ convex
2. For $m, m' \in \Delta$, $\phi(m) + \phi(m') \geq \phi(m+m')$ with equality iff m, m' in same σ

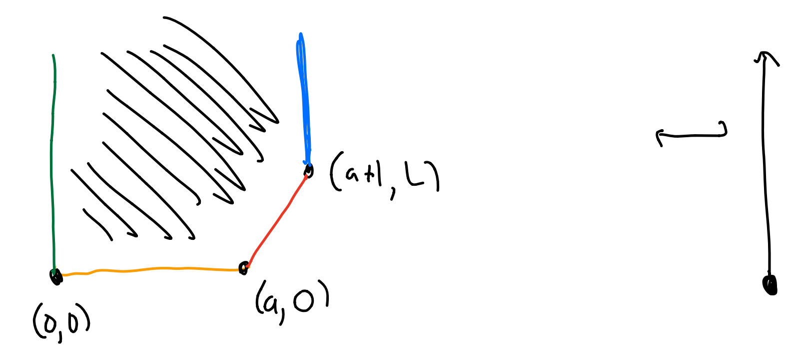
Mumford Degeneration Examples

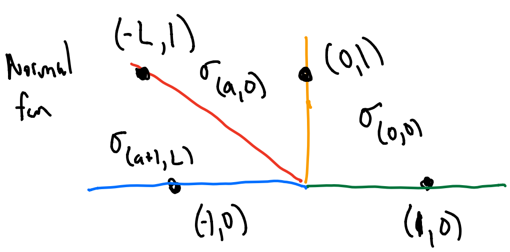
Degeneration of $\mathbb{P}^1$'s

$\Delta = [0, a+1]$ $\quad P = ([0,a], [a, a+1])$

$\phi : \Delta \to \mathbb{R}$ $\quad \phi(0) = \phi(a) = 0$ $\quad \phi(a+1) = L$

Consider $\tilde{\Delta}$ = upper convex hull of graph of ϕ





$X_{(0,0)} = \operatorname{Spec} \mathbb{C}[\chi^{(1,0)}, \chi^{(0,1)}]$ $\quad \chi^{(0,1)} \longleftarrow t$

$X_{(a,0)} = \operatorname{Spec} \mathbb{C}[\chi^{(0,1)}, \chi^{(-1,0)}, \chi^{(1,L)}]$ $\quad \longrightarrow \operatorname{Spec} \mathbb{C}[t]$

each has map like this

$X_{(a+1,L)} = \operatorname{Spec} \mathbb{C}[\chi^{(-1,0)}, \chi^{(0,1)}, \chi^{(-1,-L)}]$

$(-1,0) + (1,L) = L(0,1)$

relation

Mumford

Now let $\Delta \subseteq M_{\mathbb{R}}$ full dimensional compact lattice polyhedron

Let $\mathcal{P}$ be a poly decomp of Δ

Let $\phi : \Delta \to \mathbb{R}$ PL strictly convex WRT $\mathcal{P}$ with integral slopes

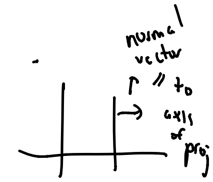
Consider $\tilde{\Delta} = \{ (m,\lambda) \in M_{\mathbb{R}} \times \mathbb{R} \mid \lambda \geq \phi(m) \}$ ($m \in \Delta$) this is lattice polyhedron.

$\mathrm{rec}(\tilde{\Delta}) = 0 \times \mathbb{R}_{\geq 0} = C$ construct $X_{\tilde{\Delta}} \to U_{\sigma_{\tilde{\Delta}}}$

$\Rightarrow X_{\tilde{\Delta}} \longrightarrow \mathbb{A}^1$

given locally by $\chi^{(0,1)} \mapsfrom t$ (map of rings)

To better understand this degeneration we study the normal fan of $\tilde{\Delta}$



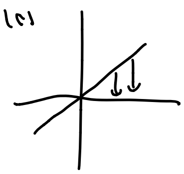
$q : M_{\mathbb{R}} \times \mathbb{R} \longrightarrow M_{\mathbb{R}}$ projection

$\breve{\Delta} \longrightarrow \Delta$

is there a supporting hyperplane which projects to supporting hyp..

Faces of $\breve{\Delta} \longrightarrow$ (nonhomeomorphically) to face here "vertical faces"

not a face "horizontal" faces



$\tilde{\Delta}$
$\cup$
$\tilde{\sigma}$ maximal horizontal face s.t. $\phi|_{p(\tilde{\sigma})}$ has slope $n_{\tilde{\sigma}} \in N$

- then the normal a ray generated by $(-n_{\tilde{\sigma}}, 1)$
- the vertical faces have normal cone contained $N_{\mathbb{R}} \times \{0\}$.

So $\chi^{(0,1)}$ vanishes on horizontal faces to order 1

doesnt vanish on vertical faces

codimension 1

$\Rightarrow$ $\pi^{-1}(0)$ = union of toric divisors corresponding to maximal horizontal faces